APPLIED MATHEMATICS I (500.303)
Exam II, FALL 2001

The following rules apply:

- Exams are due promptly at the beginning of class on Thursday, November 15, 2001. No extensions!
- PLEASE SHOW ALL WORK! In order to receive full credit, you must show ALL work and explain the reasons for any assumptions you've made. You will not be given any credit for guesses or the appearance of guesses.
- The exam must be completed **INDIVIDUALLY**. You may however consult with the instructor if you have questions. No points will be awarded if there is evidence of group work.
- You may consult the course text and any notes, handouts, or solutions from this course from FALL 2001 ONLY. All other materials are prohibited.
- You must include a cover page, signed and dated, with the following statement in your own handwriting in blue or black ink:

> "I certify that the material submitted is entirely my own work. I have adhered to all of the rules outlined for this take-home exam."

There are 9 problems, for a total of 240 points.

Part I: Stability and the Phase Plane (70 pts)

Problem 1

Let $x(t)$ and $y(t)$ represent prey and predator populations at any time $t > 0$. If we temporarily ignore the effect of encounters, the ODEs governing the populations are given by:

$$\left.\begin{aligned} \frac{dx}{dt} &= 5x - x^2, \\ \frac{dy}{dt} &= -2y. \end{aligned}\right\} \tag{1}$$

(a) **(15 pts)**. Solve for $x(t)$ and $y(t)$. Calculate $lim_{t\to\infty}x(t)$ and $lim_{t\to\infty}y(t)$. Interpret the meaning of these limits. (Note, since these are physical quantities, you should assume $x(t)$ and $y(t)$ nonnegative for all t.)

Now, we include interation effects, resulting in the new system of ODEs:

$$\left.\begin{aligned} \frac{dx}{dt} &= 5x - x^2 - xy, \\ \frac{dy}{dt} &= xy - 2y. \end{aligned}\right\} \tag{2}$$

(b) **(5 pts)**. What are the critical points of system (2)? Determine their type and stability.

(c) **(5 pts)**. Draw a phase plot for this system, and explain what your graph shows.

(d) **(5 pts)**. Denote the critical point for which both species survive as (x^*, y^*). Define the deviation from equilibrium for each population by $\bar{x}$ and $\bar{y}$, and linearize system (2) about (x^*, y^*) to obtain a linear system of differential equations in $\bar{x}$ and $\bar{y}$.

(e) **(10 pts)**. Use the initial conditions $\bar{x}(0) = 5$ and $\bar{y}(0) = 4$ to solve the linearized system exactly for $\bar{x}(t)$ and $\bar{y}(t)$. (Suggestion: Use Laplace transforms.)

(f) **(5 pts)**. Keeping in mind that $x = \bar{x} + x^*$ and $y = \bar{y} + y^*$, calculate $lim_{t\to\infty}x(t)$ and $lim_{t\to\infty}y(t)$.

Problem 2

The nonlinear system

$$\left.\begin{aligned}\frac{dx}{dt} &= -\alpha xy \\ \frac{dy}{dt} &= \alpha xy - \beta y.\end{aligned}\right\} \qquad (3)$$

$\alpha, \beta > 0$ can be used to model an epidemic. In this case, $x(t)$ denotes the number of individuals susceptible to the disease and $y(t)$ denotes the number of individuals infected. The positive constants α and β are characteristic of a particular disease.

(a) **(10 pts)**. Suppose we are able to quarantine the infected population. This means that there are no interactions. Solve for $x(t)$ and $y(t)$. Calculate $lim_{t\to\infty}x(t)$ and $lim_{t\to\infty}y(t)$. Interpret the meaning of these limits. (Note, since these are physical quantities, you should assume $x(t)$ and $y(t)$ nonnegative for all t.)

(b) **(5 pts)**. What are the critical points of this system? Determine their type and stability. (Note, the behavior of the system clearly depends on α and β.)

(c) **(5 pts)**. Determine the equation for the trajectories in the x-y coordinate system.

(d) **(5 pts)**. Consider the following three scenarios: Scenario I: $\alpha > \beta$; Scenario II: $\alpha < \beta$; Scenario III: $\alpha = \beta$. Assume that $x(0) > 1$ and $y(0) \geq 1$. Which scenario(s) will lead to an epidemic with 100% certainty? Explain your answer.

Part II: Numerical Methods (65 pts)

Problem 3

Consider the initial value problem

$$\frac{dy}{dx} = x^2 + y^2, \qquad y(0) = 1$$

on the interval $0 \le x \le 1$.

(a) **(5 pts)**. For this specific problem, what is the Euler's method formula for y_{n+1}?

The table below shows the estimated function values when $x = 0.1, 0.2, \ldots, 0.9, 1.0$ using Euler's method for $h = 0.1$, $h = 0.02$, and $h = 0.005$.

x	y with $h = 0.1$	y with $h = 0.02$	y with $h = 0.005$
0.0	1.0000	1.0000	1.0000
0.1	1.1000	1.1088	1.1108
0.2	1.2220	1.2458	1.2512
0.3	1.3753	1.4243	1.4357
0.4	1.5735	1.6658	1.6882
0.5	1.8371	2.0074	2.0512
0.6	2.1995	2.5201	2.6104
0.7	2.7193	3.3612	3.5067
0.8	3.5078	4.9601	5.5763
0.9	4.8023	8.9999	12.2060
1.0	7.1895	30.9167	1502.0943

(b) **(5 pts)**. For each h, what is the total number of function evaluations (i.e., calculations of $x^2 + y^2$) required to estimate $y(1.0)$?

(c) **(5 pts)**. What do you observe about the performance of Euler's method on this ODE?

(d) **(5 pts)**. Sketch or plot the direction field and a few solution curves for this ODE.

(e) **(5 pts)**. Based on your graph, why do you think Euler's method behaves the way it does?

Problem 4

Consider the initial value problem

$$\frac{dy}{dx} = 5y - 6e^{-x}, \qquad y(0) = 1$$

on the interval $0 \leq x \leq 4$.

(a) **(5 pts)**. What is the general solution? What is the specific solution (i.e., the one that satisfies the initial conditions?)

(b) **(5 pts)**. For this specific problem, what are the Runge-Kutta method formulas for y_{n+1}, k_1, k_2, k_3, and k_1?

The table below shows the estimated function values using the Runge-Kutta method with $h = 0.2$, $h = 0.01$, and $h = 0.05$ and the actual y values when $x = 0.4, 0.8, \ldots, 3.6, 4.0$.

x	RK y with $h = 0.2$	RK y with $h = 0.1$	RK y with $h = 0.05$	Actual y
0.0	1.0000	1.0000	1.0000	1.0000
0.4	0.6688	0.6702	0.6703	0.6703
0.8	0.4371	0.4483	0.4493	0.4493
1.2	0.2110	0.2938	0.3007	0.3012
1.6	-0.4602	0.1470	0.1980	0.2019
2.0	-4.7214	-0.2703	0.1067	0.1353
2.4	-35.5342	-2.9042	-0.1210	0.0907
2.8	-261.2502	-22.0535	-1.5037	0.0608
3.2	-1916.6939	-163.2507	-11.5186	0.0408
3.6	-14059.3547	-1205.7122	-85.3812	0.0273
4.0	-103126.5251	-8903.1267	-631.0370	0.0183

(c) **(5 pts)**. For each h, what is the total number of function evaluations (i.e., calculations of $5y - 6e^{-x}$) required to estimate $y(4.0)$?

(d) **(5 pts)**. What do you observe about the performance of Runge-Kutta on this ODE?

(e) **(5 pts)**. Sketch or plot the general solution for various values of the integration constant c. Make sure you pick a representative range of values.

(f) **(5 pts)**. Based on your graph, why do you think Runge-Kutta behaves the way it does?

<u>Problem 5</u>

(10 pts). Consider the problems we ran into when applying numerical methods to solve ODEs in problems 3 and 4 above. Suggest an approach that could be used when applying a numerical method on any ODE that might be able detect these types of problems.

Part III: Perturbation Techniques (60 pts)

Problem 6

Consider the perturbed-ODE

$$(t + \epsilon y)\frac{dy}{dt} + y = 0, \qquad 0 < \epsilon << 1, \qquad y(1) = 1$$

(a) **(5 pts)**. This first-order ODE is exact. What is the true solution, which we shall denote as $y(t;\epsilon)$?

(b) **(10 pts)**. Approximate $y(t;\epsilon)$ as $\hat{y}(t) = y_0(t) + \epsilon y_1(t)$. Derive differential equations for y_0 and y_1 along with initial conditions for each. Solve the resulting equations.

(c) **(10 pts)**. Show that

$$y(t;\epsilon)|_{\epsilon=0} = y_0 \qquad \text{and}$$

$$\frac{\partial y(t;\epsilon)}{\partial \epsilon}|_{\epsilon=0} = y_1$$

(Note, since your equation for $y(t;\epsilon)$ is given in implicit form, when you take derivatives with respect to ϵ, treat t as a constant and y as a variable.)

Problem 7

Consider the perturbed-ODE

$$\epsilon\frac{d^2y}{dt^2} + \frac{dy}{dt} + y = 0, \qquad 0 < \epsilon << 1.$$

A solution is to be obtained in the interval $0 < t < 1$ subject to the boundary conditions $y(0) = 0$, $y(1) = 1$.

(a) **(5 pts)**. Find the true solution, $y(t;\epsilon)$.

Let λ_1 and λ_2 be the roots to the characteristic equation you used to find the true solution. Note that

$$\sqrt{1+x} = 1 + \frac{1}{2}x - \frac{1}{4}x^2 + \cdots$$

(b) **(5 pts)**. Make the appropriate substitution into the above formula and approximate λ_1 and λ_2. Ignore any terms with ϵ raised to the power 2 or higher.

(c) **(5 pts)**. We will approximate $y(t;\epsilon)$ as $\hat{y}(t) = y_0(t) + \epsilon y_1(t)$. Derive differential equations for y_0 and y_1 along with boundary conditions for each.

(d) **(5 pts)**. What problem do you now observe?

(e) **(5 pts)**. Solve the system of ODEs using ONLY the boundary condition at $t = 1$.

(f) **(5 pts)**. Plot y_0, $\hat{y}(t)$, and $y(t;\epsilon)$ on the same graph. Use $\epsilon = 0.05$.

(g) **(5 pts)**. Based on your graph, why do you think the perturbation expansion fails when t is close to zero? Explain why it was "reasonable" to ignore the boundary condition at $t = 0$?

Part IV: Linear Equations with Variable Coefficients (45 pts)

Problem 8

The Cauchy-Euler equations and Constant-Coefficient equations can both be solved "algebraically" (i.e., without actual integration.)

(a) **(10 pts)**. Show that one can transform a homogeneous Cauchy-Euler equation of order n into a constant-coefficient equation of order n by setting $x = e^t$.

In class, we briefly discussed whether undetermined coefficients would work for the Cauchy-Euler equation. Turns out that it does ... in certain cases.

(b) **(10 pts)**. Describe a list of functions for which undetermined coefficients could be used to solve a nonhomogeneous Cauchy-Euler equation. Your list should be as general as possible. Explain why you think undetermined coefficients will work in these cases.

Problem 9

The Hermite equation of order n is

$$y'' - 2xy' + 2ny = 0.$$

(a) **(10 pts)**. Derive the two power series solutions

$$y_1 = 1 + \sum_{m=0}^{\infty} (-1)^m \frac{2^m n(n-2)\cdots(n-2m+2)}{(2m)!} x^{2m}$$

and

$$y_2 = x + \sum_{m=0}^{\infty} (-1)^m \frac{2^m (n-1)(n-3)\cdots(n-2m+1)}{(2m+1)!} x^{2m+1}$$

(b) **(5 pts)**. Give a convincing argument why y_1 is a polynomial if n is an even integer, while y_2 is a polynomial if n is an odd integer.

The Hermite polynomial of degree n is denoted by $H_n(x)$. It is the nth degree polynomial solution of Hermite's equation, multiplied by a suitable constant so that the coefficient of x^n is 2^n. For example, $H_0(x) = 1$ and $2^0 = 1$; $H_1(x) = 2x$ and $2^1 = 2$.

(c) **(10 pts)**. Find $H_3(x)$, $H_4(x)$, $H_5(x)$, and $H_6(x)$.