Department of Applied Mathematics and Statistics
The Johns Hopkins University

INTRODUCTORY EXAMINATION–WINTER SESSION

Wednesday, January 17, 2007

Instructions: Read carefully!

1. This **closed-book** examination consists of 20 problems (sorry, no choices), each worth 5 points. The passing grade has been set at $66\frac{2}{3}$%. Partial credit will be given as appropriate; each part of a problem will be given the same weight. If you are unable to prove a result asserted in one part of a problem, you may still use that result to help in answering a later part.

2. You have been provided with a syllabus indicating the scope of the exam. Our purpose is to test not only your knowledge, but also your ability to apply that knowledge, and to provide mathematical arguments presented in **clear, logically justified steps**. The grading will reflect that broader purpose.

3. The problems have not been grouped by topic, but there are roughly equally many mainly motivated by each of the four areas identified in the syllabus (linear algebra; real analysis; probability; discrete mathematics and operations research/optimization). Nor have the problems been arranged systematically by difficulty. If a problem directs you to use a particular method of analysis, you *must* use it in order to receive substantial credit.

4. Start your answer to each problem on a NEW sheet of paper. Write only on ONE SIDE of each sheet, and please do not write very near the margins on any sheet. Arrange the sheets in order, and write your NAME and the PROBLEM NUMBER on each sheet.

5. The examination will begin at 8:30 AM; lunch and refreshments will be provided. The exam will end just before 5:00 PM. You may leave before then, but in that case you may not return.

6. Paper will be provided, but you should bring and use writing instruments that yield marks dark enough to be read easily.

7. **No calculators of any sort are needed or permitted.**

1. Let n be a positive integer. Prove that

$$(2n)(2n-1)(2n-2)\cdots(n+1)$$

is divisible by $n!$

2. Let $(x_i, y_i),\ i = 1,2,3$ be three points in the plane. Define

$$A = \begin{bmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{bmatrix}$$

and let S be defined as the set of points (x, y) for which

$$\det \begin{bmatrix} 1 & 1 & 1 & 1 \\ x & x_1 & x_2 & x_3 \\ y & y_1 & y_2 & y_3 \\ x^2+y^2 & x_1^2+y_1^2 & x_2^2+y_2^2 & x_3^2+y_3^2 \end{bmatrix} = 0$$

Show that if $\det A \neq 0$ then S is a circle (with positive radius) that passes through all of the three points (x_i, y_i).

3. Let X and Y be independent Bernoulli random variables with parameter $p = 1/2$. Let $U = X + Y$ and $V = |X - Y|$.

   Compute the correlation between U and V. Are U and V independent?

4. Let $I \subseteq \mathbb{R}$ be an interval and let $f : I \to \mathbb{R}$ be increasing on I. Suppose that $c \in I$ is not an end point of I.

   Show that $\lim_{x \to c^-} f = \sup\{f(x) : x \in I, x < c\}$.

5. Let $A = \{a_1, a_2, \ldots, a_m\}$ be distinct "demand points" in $\mathbb{R}^n$, and let $W = \{w_1, w_2, \ldots, w_m\}$ be a corresponding set of positive numerical "weights". The Weber Plant Location Problem with these data, $P(A,W)$, calls for determining a point x in $\mathbb{R}^n$ with a minimum sum of weighted Euclidean distances from the demand points, i.e., calls for

$$\text{Minimize } f(x) := w_1|x-a_1| + w_2|x-a_2| + \cdots + w_m|x-a_m|.$$

   Show that $P(A,W)$ is a convex program, and that it has at least one optimal solution.

6. Recall that the Frobenius norm of real matrix $M = [m_{ij}]$ is given by

$$||M||_F^2 = \sum_{ij} m_{ij}^2.$$

A standard decomposition of a real square $(n \times n)$ matrix M into symmetric and antisymmetric parts is given by $M = S + A$, where $S = (M + M^T)/2$ and $A = (M - M^T)/2$.
Prove that $||M||_F^2 = ||S||_F^2 + ||A||_F^2$.

7. The probability of being dealt a full house in a single hand of poker is approximately .0014. Find an approximation for the probability that in 1000 independent hands of poker you will be dealt at least 2 full houses.

8. Assume that a, b, c, d are real numbers in $[-1, 1]$ such that $|a + c| \leq 1$ and $|b + d| \leq 1$. Show that $|ad - bc| \leq 1$.

9. Show that the simplex method will terminate in a finite number of iterations if degeneracy never occurs.

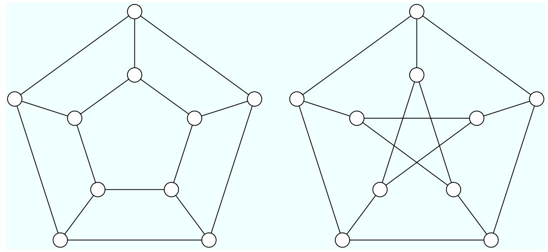
10. Let $A = \begin{bmatrix} 2 & 3 \\ 0 & 2 \end{bmatrix}$. Does there exist a matrix P such that PAP^{-1} is a diagonal matrix? If not, prove that no such matrix exists and if so, give such a matrix.

11. In a probability space, suppose that $\{E_n, n \geq 1\}$ and $\{F_n, n \geq 1\}$ are increasing sequences of events having limits E and F respectively. Show that if E_n is independent of F_n for all $n \geq 1$ then E is independent of F.

12. Let A be a $n \times n$ symmetric, nonsingular real matrix, $b \in \mathbb{R}^n$ and $\alpha \in \mathbb{R}$. Show that if $\hat{x}$ is a stationary (critical) point of the function $f(x) = x^T A x + 2b^T x + \alpha$ then

$$f(\hat{x}) = \frac{\det\left(\begin{bmatrix} A & b \\ b^T & \alpha \end{bmatrix}\right)}{\det(A)}.$$

13. Consider the two graphs in the figure. Prove that they are not isomorphic.



14. (a) Prove that a real symmetric matrix A is positive definite if and only if $A = B^T B$, where B is a real, non-singular matrix.

    (b) If a real symmetric matrix A is positive definite, show that $\det A > 0$.

15. Let X be a positive random variable with density $f(x)$. Show that

$$E(X) = \int_0^\infty P(X \geq t) dt.$$

16. Consider the space $C[0,1]$ of real-valued continuous functions on the domain $[0,1]$. For $f \in C[0,1]$, define the function Jf by indefinite integration:

$$(Jf)(x) := \int_0^x f(t)\, dt, \quad x \in [0,1].$$

    (a) Does J map $C[0,1]$ into $C[0,1]$? Is it one-to-one? Is it onto $C[0,1]$?

    (b) For which functions f does the sequence $(J^n f)$ converge uniformly to a limit function belonging to $C[0,1]$? Here J^n denotes the nth iterate of J.

17. Let T be a tree and let u and v be distinct vertices of T that are not adjacent. Prove that T contains an edge e such that $(T - e) + uv$ is also a tree.

    Note: The notation $(T - e) + uv$ means the graph formed from T by deleting the edge e and then adding in the edge uv.

18. Suppose that A is an $n \times n$ matrix with distinct eigenvalues λ_i, $i = 1, .., n$. If $e(t)$ is the n-vector with components $e_i(t) = e^{\lambda_i t}$ and if V is the Vandermonde matrix with entries $V_{ij} = \lambda_i^{j-1}$, then prove that

$$e^{At} = \sum_{j=1}^{n} d_j(t) A^{j-1}$$

where the coefficients are the components of the n-vector

$$d(t) = V^{-1} e(t).$$

Hint: Use the Cayley-Hamilton theorem.

19. Show that if X_1 and X_2 are independent Poisson random variables with parameters θ_1 and θ_2, respectively, then $X_1 + X_2$ has a Poisson distribution with parameter $\theta_1 + \theta_2$.

20. Let $f : \mathbb{R}^p \to \mathbb{R}$ be differentiable at c.

(a) Express the directional derivative of f at c in the direction of a given unit vector $w = (w_1, \ldots, w_p)$. Do this in terms of the partial derivatives of f.

(b) Show that there is a direction in which the directional derivative is maximum and that this direction is uniquely determined if at least one of the partial derivatives is nonzero at c.