

Department of Applied Mathematics and Statistics  
The Johns Hopkins University

INTRODUCTORY EXAMINATION—FALL SESSION

Wednesday, August 30, 2006

**Instructions: Read carefully!**

1. This **closed-book** examination consists of 20 problems (sorry, no choices), each worth 5 points. The passing grade has been set at  $66\frac{2}{3}\%$ . Partial credit will be given as appropriate; each part of a problem will be given the same weight. If you are unable to prove a result asserted in one part of a problem, you may still use that result to help in answering a later part.
2. You have been provided with a syllabus indicating the scope of the exam. Our purpose is to test not only your knowledge, but also your ability to apply that knowledge, and to provide mathematical arguments presented in **clear, logically justified steps**. The grading will reflect that broader purpose.
3. The problems have not been grouped by topic, but there are roughly equally many mainly motivated by each of the four areas identified in the syllabus (linear algebra; real analysis; probability; discrete mathematics and operations research/optimization). Nor have the problems been arranged systematically by difficulty. If a problem directs you to use a particular method of analysis, you *must* use it in order to receive substantial credit.
4. Start your answer to each problem on a **NEW** sheet of paper. Write only on **ONE SIDE** of each sheet, and please do not write very near the margins on any sheet. Arrange the sheets in order, and write your **NAME** and the **PROBLEM NUMBER** on each sheet.
5. The examination will begin at 8:30 AM; lunch and refreshments will be provided. The exam will end just before 5:00 PM. You may leave before then, but in that case you may not return.
6. Paper will be provided, but you should bring and use writing instruments that yield marks dark enough to be read easily.
7. **No calculators of any sort are needed or permitted.**

1. An  $n \times n$  matrix  $P = (p_{ij})$  is said to be *stochastic* if all of its entries are nonnegative and the sum of the entries in each row is 1.

If  $P$  is an  $n \times n$  stochastic matrix, prove the existence of a nonzero nonnegative solution to the system of equations

$$\sum_{i=1}^n y_i p_{ij} = y_j, \quad j = 1, \dots, n.$$

(Hint: One approach involves introducing a related linear program.)

*Solution: Via Farkas' Lemma:*

Farkas' Lemma says that for a given  $m \times n$  matrix  $A$ , and an  $n$ -vector  $b$  the following statements are equivalent:

(F1) There exists  $x \geq 0$  such that  $Ax = b$ .

(F2)  $y^T A \geq 0$  implies  $y^T b \geq 0$  for all  $m$ -vectors  $y$ .

We want to prove the existence of a nonzero  $x$  such that

$$(P^T - I)x = 0.$$

Were such a solution to exist we could normalize its entries so that

$$1^T x = 1,$$

so in matrix terms it suffices to show there exists a nonnegative  $x$  such that

$$\begin{bmatrix} P^T - I \\ 1^T \end{bmatrix} x = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

To use Farkas' Lemma, take  $A = \begin{bmatrix} P^T - I \\ 1^T \end{bmatrix}$  and  $b = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ .

We proceed to show that (F2) holds, so fix  $y = [v^T, c]^T$  where  $v$  is an  $n$ -vector and  $c$  is a scalar. Direct calculation yields

$$y^T A = [v^T, c] \begin{bmatrix} P^T - I \\ 1^T \end{bmatrix} = (v^T (P^T - I) + c1^T) = v^T P^T - v^T + c1^T.$$

thus

$$(y^T A)_i = (Pv)_i - v_i + c.$$

Note also that  $y^T b = c$ .

If  $y^T A \geq 0$  we have

$$c \geq v_i - (Pv)_i, \quad \forall i.$$

Taking  $I = \arg \max_i v_i$  and using the fact that  $P$  is stochastic, we have  $v_I \geq (Pv)_I$  so we conclude that  $y^T b = c \geq 0$ .

*Via linear programming:*

If such a solution exists we can normalize it so that its entries sum to one. Thus, the desired result is equivalent to the feasibility (existence of a feasible solution to) the linear program

$$\begin{aligned} \text{(P)} \quad & \max Z = \sum_{j=1}^n 0 \cdot y_j \\ & \text{subject to} \\ & \sum_{j \neq i} p_{ji} y_j + (p_{ii} - 1) y_i = 0 \quad i = 1, \dots, n \\ \text{(*)} \quad & \sum_{j=1}^n y_j = 1 \\ & y_j \geq 0, j = 1, \dots, n. \end{aligned}$$

We proceed to form the dual linear program with unsigned variables  $v_1, \dots, v_n$  corresponding to the first  $n$  constraints of (P), and an unsigned variable  $v$  corresponding to the constraint (\*).

$$\begin{aligned} \text{(D)} \quad & \min Z' = \sum_{i=1}^n 0 \cdot v_i + 1 \cdot v = v \\ & \text{subject to} \\ & \sum_{i \neq j} p_{ji} v_i + (p_{jj} - 1) v_j \geq 0 \quad j = 1, \dots, n \end{aligned}$$

Here (D) is certainly feasible, since its constraints are satisfied by taking  $v = 0$  and all  $v_i = 0$ . Also,  $Z'$  is bounded below, since because  $P$  is stochastic we have

$$p_{jj} - 1 = - \sum_{i \neq j} p_{ji},$$

permitting the  $j$ -th constraint of (D) to be written as

$$v \geq \sum_{i \neq j} p_{ji} (v_j - v_i), \text{ for all } j,$$

so that for any  $(v_1, \dots, v_n)$  we can choose  $J \in \arg \max v_i$  and then the  $J$ -th constraint of (D) (since all  $p_{ji} \geq 0$  will imply  $v \geq 0$ . That is,  $Z' \geq 0$ . By the duality theorem of linear programming, since (D) is bounded and feasible, (P) must be feasible, the desired result.

2. Which (if either) of these sums converges?

$$\sum_{n=2}^{\infty} \frac{1}{n \log n} \quad \text{and} \quad \sum_{n=2}^{\infty} \frac{1}{n \log^2 n}$$

Justify your answer.

For this problem, we assume the logarithms are base  $e$ , but does that affect your answer?

*Solution:* We apply the integral test to both.

For the first, note that

$$\int \frac{dx}{x \log x} = \log \log x$$

(by an easy substitution  $u = \log x$ ) which diverges as  $x \rightarrow \infty$ . Therefore the first sum diverges.

For the second we have

$$\int \frac{dx}{x \log^2 x} = \frac{-1}{\log x}$$

(also by substituting  $u = \log x$ ). This converges as  $x \rightarrow \infty$ , and so does the corresponding sum.

The convergence of the sums does not depend on the base of the logarithm.

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3. Let  $O$  be an  $n \times n$  real orthogonal matrix, i.e. such that

$$O^T O = O O^T = I.$$

(a) Prove that  $\det O = \pm 1$ .

(b) If  $n$  is odd, prove that  $O\mathbf{x} = (\det O)\mathbf{x}$  for some nonzero  $\mathbf{x} \in \mathbb{R}^n$ .

*Solution:* (a)  $|\det O|^2 = \det(O^T O) = 1$ .

(b) We can take  $\det O = 1$ , since otherwise we consider  $-O$ . Then note that

$$\det(O - I) = \det(O - O O^T) = \det O \cdot \det(I - O^T) = 1 \cdot \det(I - O) = (-1)^n \det(O - I).$$

If  $n$  is odd, then  $\det(O - I) = 0$  and  $O$  has an eigenvector with eigenvalue 1.

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4. Let  $G$  be a simple graph (no loops or multiple edges) with vertex set  $\{v_1, v_2, v_3, v_4, v_5\}$ . The degree sequence of  $G$  is a list of the degrees of the vertices in the graph, i.e.,  $d(v_1), d(v_2), d(v_3), d(v_4), d(v_5)$ , often listed in numerical order.

Only one of the following three sequences can possibly be the degree sequence of  $G$ .

(a) 1, 2, 2, 3, 3

(b) 1, 1, 3, 3, 4

(c) 2, 2, 2, 3, 3

Prove that two of these sequences cannot be the degree sequence of  $G$  and then demonstrate that the third sequence is feasible by drawing a picture of  $G$ .

*Solution:*

(a) This sequence is not realizable because it sums to 11 and the sum of the degrees of the vertices in a graph cannot be odd.

(b) This sequence is not realizable. Taking

$$[d(v_1), d(v_2), d(v_3), d(v_4), d(v_5)] = [1, 1, 3, 3, 4]$$

we see that  $v_5$  is adjacent to all vertices implying that  $v_1, v_2$  are adjacent only to  $v_5$ . Hence  $v_3$  and  $v_4$  may be adjacent only to each other and to  $v_5$ , limiting their degree to at most 2. Therefore  $d(v_3) = d(v_4) = 3$  is impossible.

(c) Here's a drawing of the graph that realizes the sequence 2, 2, 2, 3, 3:

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5. Let  $X$  be a  $2 \times 4$  real matrix. We calculate

$$X^T X = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 4 & 2 & 4 \\ 1 & 2 & 2 & 3 \\ 1 & 4 & 3 & ? \end{bmatrix}.$$

Find (with proof) the missing entry (denoted with a question mark).

*Solution:* Note that because  $X$  is a  $2 \times 4$  matrix, the product  $X^T X$  can have rank at most 2. The first two columns of  $X^T X$  are linearly independent, and so that last column must be linearly dependent on the first two, i.e.,

$$\begin{bmatrix} 1 \\ 4 \\ 3 \\ ? \end{bmatrix} = a \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix} + b \begin{bmatrix} 0 \\ 4 \\ 2 \\ 4 \end{bmatrix}$$

for some scalars  $a$  and  $b$ . A glance at the first two rows gives  $a = b = 1$  and so the missing entry is  $1 + 4 = 5$ .

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6. Here  $\{f_n : n = 1, 2, \dots\}$  is a sequence of continuous real-valued functions defined on a common interval  $I = [a, b]$  such that, for each  $x \in I$ , the sequence  $f_n(x)$  converges to a limit, denoted  $f(x)$ . Prove that if the convergence is uniform, then the limit-function  $f$  is also continuous.

*Solution:* We show that for any  $x$  in  $I$ ,  $f$  is continuous at  $x$ . As always, consider any  $\varepsilon > 0$ . Since the convergence is uniform, there is a positive integer  $N$  such that for all  $n > N$  and all  $y \in I$ , we have

$$|f_n(y) - f(y)| < \varepsilon/3. \quad (1)$$

Fix such an  $n$ . Since  $f_n$  is continuous at  $x$ , there is a  $\delta > 0$  for which, for all  $z \in I$ ,

$$|z - x| < \delta \Rightarrow |f_n(z) - f_n(x)| < \varepsilon/3. \quad (2)$$

In particular, consider any  $z \in I$  with  $|z - x| < \delta$ . Besides (??), by (??) with  $y = z$  and with  $y = x$ , we have

$$|f(z) - f_n(z)| < \varepsilon/3 \quad \text{and} \quad |f_n(x) - f(x)| < \varepsilon/3.$$

So

$$\begin{aligned} |f(z) - f(x)| &\leq |f(z) - f_n(z)| + |f_n(z) - f_n(x)| + |f_n(x) - f(x)| \\ &< (\varepsilon/3) + (\varepsilon/3) + (\varepsilon/3) = \varepsilon. \end{aligned}$$

We have shown that for any  $\varepsilon > 0$ , there is a  $\delta > 0$  such that  $|z - x| < \delta \Rightarrow |f(z) - f(x)| < \varepsilon$ . So  $f$  is indeed continuous at  $x$ .

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7. Let  $A$  be an  $n \times n$  matrix and assume that  $A$  has  $n$  distinct eigenvalues. Let  $\mathcal{V}$  be the set of all  $n \times n$  matrices  $B$  such that  $AB = BA$ . Show that  $\mathcal{V}$  is a vector space and compute its dimension. (Hint: Show that  $B$  maps eigenvectors of  $A$  to eigenvectors of  $A$ .)

*Solution:*  $\mathcal{V}$  contains 0, and is closed under linear combinations: if  $a, a' \in \mathbb{R}$ , and  $B, B' \in \mathcal{V}$ , then  $A(aB + a'B') = aAB + a'AB' = aBA + a'B'A = (aB + a'B')A$ .

Let  $\lambda_1, \dots, \lambda_n$  be the eigenvalues of  $A$ , and  $e_k$  an eigenvector for  $\lambda_k$ . Since the  $\lambda_k$ 's are distinct,  $\{e_1, \dots, e_n\}$  is a basis of  $\mathbb{R}^n$ . We have  $ABe_k = BAe_k = \lambda_k Be_k$ . Consequently,  $Be_k$  is an eigenvector of  $A$  with the same eigenvalue  $\lambda_k$ . This implies that, for some  $\alpha_k$ ,  $Be_k = \alpha_k e_k$ , which uniquely defines  $B$  since  $\{e_1, \dots, e_n\}$  is a basis. Conversely, any  $B$  which is diagonal in the basis  $\{e_1, \dots, e_n\}$  commutes with  $A$ . This describes the set  $\mathcal{V}$  which therefore has dimension  $n$ .

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8. Let  $a(n, x) = \lfloor \frac{n}{2} + \frac{x}{2}\sqrt{n} \rfloor$  where  $n$  is a positive integer,  $x$  is real and  $\lfloor y \rfloor$  denotes the greatest integer less than or equal to  $y$ . Evaluate

$$\lim_{n \rightarrow \infty} 2^{-n} \sum_{k=0}^{a(n,x)} \binom{n}{k}.$$

[Hint: Use the Central Limit Theorem.]

*Solution:* Let  $X_1, X_2, \dots$  be independent and identically distributed Bernoulli random variables with  $P(X_i = 1) = .5$  and let  $S_n = X_1 + \dots + X_n$ . By the CLT:

$$\lim_{n \rightarrow \infty} P\left(\frac{S_n - \frac{n}{2}}{\sqrt{n}/2} \leq x\right) = \Phi(x)$$

where  $\Phi(x)$  is the standard normal distribution function. Since

$$\begin{aligned} P\left(\frac{S_n - \frac{n}{2}}{\sqrt{n}/2} \leq x\right) &= P\left(S_n \leq \frac{n}{2} + \frac{x\sqrt{n}}{2}\right) \\ &= \sum_{k=0}^{a(n,x)} \binom{n}{k} \left(\frac{1}{2}\right)^n \end{aligned}$$

the limit in question is  $\Phi(x)$  for all  $x$

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9. The number of births per day in a small town hospital has the following distribution:

# births	0	1	2	3	4
probability	.25	.45	.14	.11	.05

Assume that each baby has probability 1/2 to be a girl. What is the most likely number of births in a day if it is known that exactly two girls are born ?

*Solution:* Let  $X$  be the number of births on a given day and let  $Y$  denote the number of girls born that day. We must compute  $P(X = x|Y = 2)$ :

$$P(X = x|Y = 2) = P(X = x, Y = 2)/P(Y = 2)$$

Since we need only the most likely  $x$ , we needn't calculate the term  $P(Y = 2)$ . We have (for  $x > 1$ ):

$$\begin{aligned} P(X = x, Y = 2) &= P(Y = 2|X = x)P(X = x) \\ &= \binom{x}{2} \times 0.5^2 \times 0.5^{x-2}P(X = x) \\ &= \binom{x}{2} \times 0.5^x P(X = x) \end{aligned}$$

For  $x = 2, 3, 4$ , this yields values  $0.14/4, 3(0.11)/8$  and  $6(0.05)/16$ , respectively. The maximum is at  $x = 3$ .

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10. Let  $a_1, a_2, \dots, a_n$  be positive numbers. Prove that

$$\sum_{i=1}^n a_i \sum_{j=1}^n 1/a_j \geq n^2$$

and equality holds if and only if  $a_1 = a_2 = \dots = a_n$ .

*Solution:* Let  $x_i = \sqrt{a_i}$  and  $y_i = 1/\sqrt{a_i}$ . Then by the Cauchy-Schwartz inequality we see that

$$n^2 = \left\{ \sum_{i=1}^n 1 \right\}^2 = \left\{ \sum_{i=1}^n x_i y_i \right\}^2 \leq \left\{ \sum_{i=1}^n x_i^2 \right\} \left\{ \sum_{i=1}^n y_i^2 \right\} = \sum_{i=1}^n a_i \sum_{j=1}^n 1/a_j$$

Furthermore, equality holds if and only if

$$(a_1, \dots, a_n) = c(1/a_1, \dots, 1/a_n)$$

for some positive constant  $c$ , since the  $a_i$  are positive. But this condition gives  $a_i^2 = c$  so  $a_i = \sqrt{c}$ , for  $i = 1, \dots, n$ .

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11. Let  $X$  have distribution function  $F_X$ , with probability density function  $f_X$ .

Let  $Y$  have distribution function  $F_Y$ , with probability density function  $f_Y$ .

Assume  $X$  and  $Y$  are independent.

Recall that  $X <^{st} Y$  (read “ $X$  is stochastically smaller than  $Y$ ”) means that  $F_X(z) \geq F_Y(z)$  for all  $z$  with strict inequality for at least one  $z$ .

Prove that  $X <^{st} Y \implies P[X < Y] > 1/2$ .

*Solution:*

$$\begin{aligned} P[X < Y] &= \int P[X < Y | Y = y] f_Y(y) dy \\ &= \int F_X(y) f_Y(y) dy \\ &\geq \int F_Y(y) f_Y(y) dy \\ &= E[F_Y[Y]] \\ &= E[\text{Uniform}(0, 1)] \\ &= 1/2. \end{aligned}$$

Strict inequality follows from (absolute) continuity of  $F_X$  and  $F_Y$ .

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12. (a) For real numbers  $x_1, \dots, x_n$ , find the maximum of

$$f(x_1, x_2, \dots, x_n) := (x_1 x_2 \cdots x_n)^2,$$

subject to the constraint

$$x_1^2 + x_2^2 + \cdots + x_n^2 = 1.$$

Do *not* invoke the result of part (b).

- (b) Show that the geometric mean of a collection of nonnegative real numbers  $\{a_1, \dots, a_n\}$  does not exceed their arithmetic mean; that is,

$$(a_1 a_2 \cdots a_n)^{1/n} \leq \frac{1}{n}(a_1 + a_2 + \cdots + a_n).$$

*Solution:*

- (a) It is probably easiest to use a Lagrange multiplier  $\lambda$ . We then have the system

$$\frac{2(x_1 x_2 \cdots x_n)^2}{x_i} = 2x_i \lambda, \quad i = 1, 2, \dots, n.$$

In particular, this system assigns a common value to every  $|x_i|$ ; that common value must be  $1/\sqrt{n}$  to satisfy the constraint. We then find that the maximum value is  $n^{-n}$ . (A really complete solution should discuss stationarity vs. optimality.)

- (b) If the right side vanishes, then the result is trivial. If not, then the numbers  $x_i := \sqrt{a_i} / (a_1 + a_2 + \cdots + a_n)^{1/2}$  are well defined and have squares summing to unity. Applying the result of part (a),

$$\frac{a_1 a_2 \cdots a_n}{(a_1 + a_2 + \cdots + a_n)^n} \leq n^{-n},$$

which then can be rearranged to give the desired result.

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13. Let  $\hat{x}$  be a feasible point to the following linear program:

$$\begin{aligned} \min \quad & c^T x \\ \text{s.t.} \quad & a_i^T x \geq \beta_i \quad i = 1, \dots, m. \end{aligned}$$

Let  $I(\hat{x}) = \{i : a_i^T \hat{x} = \beta_i\}$ . Show that if

$$c = \sum_{i \in I(\hat{x})} \lambda_i a_i \quad \text{and} \quad \lambda_i \geq 0,$$

then  $\hat{x}$  is optimal.

*Solution:* Let  $y$  be another feasible point, we want to show that  $c^T \hat{x} \leq c^T y$ .

$$\begin{aligned} c^T(y - \hat{x}) &= \sum_{i \in I(\hat{x})} \lambda_i a_i^T (y - \hat{x}) \\ &= \sum_{i \in I(\hat{x})} \lambda_i (a_i^T y - \beta_i) . \\ &\geq 0. \end{aligned}$$

The last inequality follows from that all  $\lambda \geq 0$  and  $y$  is a feasible point.

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14. Let  $B(r)$  denote the ball of radius  $r$  centered at the origin (i.e., the set of points whose Euclidean distance from the origin is less than or equal to  $r$ ) and let  $X$  be a three-dimensional random vector uniformly distributed over  $B(r)$  (i.e., the result of choosing a point in  $B(r)$  at random). Find the mean of  $\|X\|$ , the distance from  $X$  to the origin.

*Solution:*

$$\begin{aligned} E\|X\| &= \int \int \int_{B(r)} \sqrt{(x^2 + y^2 + z^2)} \frac{1}{\text{vol}(B(r))} dx dy dz \\ &= \left(\frac{4}{3}\pi r^3\right)^{-1} \int_0^{2\pi} \int_0^\pi \int_0^r \rho \rho^2 \sin \phi d\rho d\phi d\theta \\ &= \left(\frac{4}{3}\pi r^3\right)^{-1} 2\pi \int_0^r \rho^3 d\rho \int_0^\pi \sin \phi d\phi \\ &= \frac{3r}{4}. \end{aligned}$$


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15. Consider the graph in the accompanying figure. There are many shortest paths from the lower left corner to the upper right corner. How many of these avoid the two vertices that are colored black? One such path is highlighted.

*Solution:* Call the lower left vertex  $a$ , the upper right vertex  $b$ , the lower left black vertex  $x$ , and the upper right black vertex  $y$ .

- There are  $\binom{12}{6}$  shortest  $(a, b)$ -paths.
- There are  $\binom{4}{2} \binom{8}{4}$  shortest  $(a, b)$ -paths that include vertex  $x$ .
- There are also  $\binom{4}{2} \binom{8}{4}$  shortest  $(a, b)$ -paths that include vertex  $y$ .

- There are  $\binom{4}{2}\binom{4}{2}\binom{4}{2}$  shortest  $(a, b)$ -paths that include both  $x$  and  $y$ .

Therefore, by inclusion-exclusion, there are

$$\binom{12}{6} - 2\binom{4}{2}\binom{8}{4} + \binom{4}{2}^3$$

shortest  $(a, b)$ -paths that avoid both  $x$  and  $y$ .

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16. Given  $n$  numbers  $x_1, \dots, x_n$ , the *Vandermonde matrix*  $V = V(x_1, \dots, x_n)$  is, by definition, the matrix  $V = [v_{ij}]$  with

$$v_{ij} = x_j^{i-1}.$$

- (a) For  $n = 2$ , verify that

$$\det V = \prod_{1 \leq i < j \leq n} (x_j - x_i).$$

- (b) Prove the formula in part (a) for all  $n \geq 1$ . [HINT: The solution is simpler if you do *not* use induction or any explicit Laplace expansion of the determinant, but rather think about the determinant as a polynomial.]

*Solution:*

- (a) When  $n = 2$ , we have

$$\det V = v_{11}v_{22} - v_{12}v_{21} = x_1^0x_2^1 - x_2^0x_1^1 = x_2 - x_1 = \prod_{1 \leq i < j \leq 2} (x_j - x_i).$$

- (b) Here is a simple solution. The determinant is a polynomial in the variables  $x_1, \dots, x_n$  of total degree

$$0 + 1 + \dots + (n - 1) = \binom{n}{2}.$$

It vanishes if  $x_i = x_j$  for some  $i < j$ , so

$$\det V = c \prod_{1 \leq i < j \leq n} (x_j - x_i)$$

for some constant  $c$ . By equating the coefficients of  $x_1^0x_2^1 \dots x_n^{n-1}$  in this last equation we find  $c = 1$ , and the formula is established.

Now here is a harder, more brute-force solution, by induction. Subtract column 1 from columns  $2, \dots, n$ . Then subtract  $x_1$  times row  $k - 1$  from row  $k$ , for  $k = n, n - 1, \dots, 2$  (in that order). We now factor  $x_k - x_1$  out of columns  $k = 2, \dots, n$ , obtaining  $(x_2 - x_1) \dots (x_n - x_1)$  times a Vandermonde determinant of order  $n - 1$ , so the calculation continues by induction.

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17. Determine whether or not the vector  $x=(1,0,1,0)$  is an optimal solution of the following linear program:

$$\begin{array}{ll}
 \min & -x_1 + 2x_2 - x_3 - x_4 \\
 \text{s.t.} & x_1 + x_2 - x_3 + 2x_4 \geq -2 \\
 & x_1 + 2x_2 - x_3 + x_4 = 0 \\
 & -x_1 - x_2 - x_3 - x_4 \geq -2 \\
 & x_1, x_2, x_3 \geq 0 \quad x_4 \text{ unrestricted.}
 \end{array}$$

*Solution:* We first consider the dual problem and find a corresponding dual solution to  $x = (1, 0, 1, 0)$ .

$$\begin{array}{ll}
 \min & -2w_1 \quad -2w_3 \\
 \text{s.t.} & w_1 + w_2 - w_3 \leq -1 \\
 & w_1 + 2w_2 - w_3 \leq 2 \\
 & -w_1 - w_2 - w_3 \leq -1 \\
 & 2w_1 + w_2 - w_3 = -1 \\
 & w_1, w_3 \geq 0 \quad w_2 \text{ unrestricted.}
 \end{array}$$

By  $x_1 > 0$  and  $x_3 > 0$ , we have the first and the third constraints of the dual as equalities:  $w_1 + w_2 - w_3 = -1$  and  $-w_1 - w_2 - w_3 = -1$ . The fourth constraint  $2w_1 + w_2 - w_3 = -1$  is another equality. Solve the system to get the dual solution  $w = (0, 0, 1)$ . Then we check (1)  $x$  is feasible to the original problem, (2)  $w$  is feasible to the dual, (3) the two objective function values have the same value  $-2$ , therefore  $x$  is an optimal solution.

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18. A primitive model for heat conduction leads to the equation

$$\begin{bmatrix} u_n \\ v_n \end{bmatrix} = A^n \begin{bmatrix} u_0 \\ v_0 \end{bmatrix}, n = 1, 2, \dots, A = \begin{bmatrix} 7/9 & 1/9 \\ 1/9 & 7/9 \end{bmatrix},$$

where  $u_n$  and  $v_n$  represent temperatures at times  $n = 0, 1, \dots$ . Find the limits  $c_u$  and  $c_v$ , where  $u_n \rightarrow c_u$  and  $v_n \rightarrow c_v$  as  $n \rightarrow \infty$ .

Proceed as follows:

- Find the eigenvalues of  $A$ .
- Represent  $A^n$  as  $CDC^{-1}$ , where  $D$  is a diagonal matrix.
- Use this representation of  $A^n$  to find the limits  $c_u$  and  $c_v$ .

*Solution:* (a)  $A$  has eigenvalues  $8/9$  and  $2/3$ .

(b)  $A = CBC^{-1}$  where  $B = \text{diag}(8/9, 2/3)$ . Therefore,

$$A^2 = (CBC^{-1})(CBC^{-1}) = CB^2C^{-1},$$

$$A^3 = (CBC^{-1})(CB^2C^{-1}) = CB^3C^{-1},$$

and, in general,

$$A^n = CB^nC^{-1}.$$

Since  $B$  is a diagonal matrix,  $B^n = \text{diag}((8/9)^n, (2/3)^n)$ .

(c) Since  $(8/9)^n \rightarrow 0$  and  $(2/3)^n \rightarrow 0$  when  $n \rightarrow \infty$ , the elements of  $B^n$  approach 0 as limit when  $n \rightarrow \infty$ . Since  $C$  and  $C^{-1}$  are fixed, so do the elements of  $A^n = CB^nC^{-1}$  approach 0 as limit when  $n \rightarrow \infty$ . Therefore,  $u_n \rightarrow 0$  and  $v_n \rightarrow 0$  as  $n \rightarrow \infty$ . That is,  $c_u = c_v = 0$ .

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19. For a set of randomly chosen people, let  $E_{i,j}$  denote the event that persons  $i$  and  $j$  have the same birthday. (Assume that each person is equally likely to have any of the 365 days of the year as his or her birthday, and that different persons' birthdays are independent.)

(a) Find  $P[E_{3,4}|E_{1,2}]$ .

(b) Find  $P[E_{1,3}|E_{1,2}]$ .

(c) Find  $P[E_{2,3}|E_{1,2} \cap E_{1,3}]$ .

(d) What can you conclude from the previous three parts about the independence of the events  $\{E_{i,j}\}$ ?

*Solution:* (a)  $1/365$ , since the two events are independent.

(b)  $1/365$ , since the probability that the birthday of person 3 matches the common birthday of 1 and 2 is  $1/365$ .

(c) 1, since the birthdays of persons 2 and 3 must both match that of person 1.

(d) The events  $\{E_{i,j}\}$  are pairwise independent, but not mutually independent.

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20. If  $\{x_n\}$  is a sequence of real numbers, then its *limit infimum* is defined by

$$x_* = \lim_{n \rightarrow \infty} \inf_{k \geq n} x_k$$

Given such a sequence, prove that there is a subsequence  $\{x_{k_n}\}$  such that  $\lim_{n \rightarrow \infty} x_{k_n} = x_*$  but no subsequence such that  $\lim_{n \rightarrow \infty} x_{k_n} = y_* < x_*$ .

*Solution:* For each  $n$  there is a  $k_n \geq n$  such that  $0 \leq x_{k_n} - \inf_{k \geq n} x_k < 1/n$ . Thus,  $\lim_{n \rightarrow \infty} x_{k_n} = \lim_{n \rightarrow \infty} \inf_{k \geq n} x_k = x_*$ .

Suppose that a subsequence exists so that  $\lim_{n \rightarrow \infty} x_{k_n} = y_* < x_*$ . Then since  $\inf_{k \geq k_n} x_k \leq x_{k_n}$ ,  $x_* = \lim_{n \rightarrow \infty} \inf_{k \geq k_n} x_k \leq y_*$ , a contradiction.

