Department of Applied Mathematics and Statistics
The Johns Hopkins University

Introductory Examination–Fall session

Wednesday, August 31st, 2005

Instructions: Read carefully!

1. This **closed-book** examination consists of 20 problems (sorry, no choices), each worth 5 points. The passing grade has been set at $66\frac{2}{3}\%$. Partial credit will be given as appropriate; each part of a problem will be given the same weight. If you are unable to prove a result asserted in one part of a problem, you may still use that result to help in answering a later part.
2. You have been provided with a syllabus indicating the scope of the exam. Our purpose is to test not only your knowledge, but also your ability to apply that knowledge, and to provide mathematical arguments presented in **clear, logically justified steps**. The grading will reflect that broader purpose.
3. The problems have not been grouped by topic, but there are roughly equally many mainly motivated by each of the four areas identified in the syllabus (linear algebra; real analysis; probability; discrete mathematics and operations research/optimization). Nor have the problems been arranged systematically by difficulty. If a problem directs you to use a particular method of analysis, you *must* use it in order to receive substantial credit.
4. Start your answer to each problem on a NEW sheet of paper. Write only on ONE SIDE of each sheet, and please do not write very near the margins on any sheet. Arrange the sheets in order, and write your NAME and the PROBLEM NUMBER on each sheet.
5. The examination will begin at 8:30 AM; lunch and refreshments will be provided. The exam will end just before 5:00 PM. You may leave before then, but in that case you may not return.
6. Paper will be provided, but you should bring and use writing instruments that yield marks dark enough to be read easily.
7. **No calculators of any sort are needed or permitted.**

Exercise 1. Suppose that A is an m by n matrix, and B is an n by s matrix. We denote by C^T the transpose of a matrix C.

Prove that $(AB)^T = B^T A^T$.

Exercise 2. Let A be an $n \times n$ matrix. Show that the rank of A equals 1 if and only if A is of the form ab^T for some choice of nonzero column n-vectors a and b.

Exercise 3. Let $A = \{a_1, a_2, \ldots, a_m\}$ be distinct "demand points" in $\mathbb{R}^n$, and let $W = \{w_1, w_2, \ldots, w_m\}$ be a corresponding set of positive numerical "weights". The Weber Plant Location Problem with these data, $P(A, W)$, calls for determining a point x in $\mathbb{R}^n$ with a minimum sum of weighted Euclidean distances from the demand points, i.e., calls for

$$\text{Minimize } f(x) := w_1|x - a_1| + w_2|x - a_2| + \cdots + w_m|x - a_m|.$$

Show that $P(A, W)$ is a convex program, and that it has at least one solution.

Exercise 4. Let X_1 and X_2 have the joint probability density function

$$\begin{aligned} f(x_1, x_2) &= 2 \text{ for } 0 < x_1 < x_2 < 1, \\ &= 0 \text{ elsewhere.} \end{aligned}$$

Let $x_2 \in (0, 1)$. Find the conditional mean of X_1, given $X_2 = x_2$.

Exercise 5. Determine whether each of the following two integrals converges or diverges:

$$\int_0^1 \int_0^1 (x^2 + y^2)^{-1/3} dxdy \quad \textit{and} \quad \int_0^1 \int_0^1 (x^2 + y^2)^{-4/3} dxdy$$

Exercise 6. Let X and Y be independent standard normal random variables. Find the joint distribution of the pair (U, V) where $U = X$ and $V = \frac{Y}{X}$. Use this to show that V has the Cauchy density

$$f(v) = \left(\pi(1 + v^2)\right)^{-1}, \quad -\infty < v < \infty.$$

Exercise 7. Let $f(n)$ denote the number of ways of dividing a strip of n unit squares into single squares and dominoes, with $f(0) = 1$ by convention. (A "domino" is a strip consisting of two adjacent squares.) For example, $f(4) = 5$, since

$$4 = 1+1+1+1 = 1+1+2 = 1+2+1 = 2+1+1 = 2+2$$

gives all the partitions of 4 into 1's and 2's. Prove the general relation

$$f(0) + f(1) + \cdots + f(n) = f(n+2) - 1.$$

Exercise 8. A collection of lines $\mathcal{L}$ in $\mathbb{R}^2$ is said to be in *general position* if

(i) each pair of lines in $\mathcal{L}$ intersect in a single point, and

(ii) the intersection of any subcollection of 3 lines in $\mathcal{L}$ is empty.

Any collection of lines $\mathcal{L}$ in general position breaks up $\mathbb{R}^2$ into a certain number of *pieces.* (Formally, these pieces are the connected open sets whose union is $\mathbb{R}^2 \setminus \bigcup_{i=1}^n L_i$.) Let $F(\mathcal{L})$ denote the *number* of these pieces. So, for example, if $\mathcal{L}$ consists of only one line, then $F(\mathcal{L}) = 2$.

(a) Show that $F(\mathcal{L})$ depends only on the number of lines in $\mathcal{L}$ so that we can define G_m to be the common value of $F(\mathcal{L})$ for all $\mathcal{L}$ in general position with $|\mathcal{L}| = m$. Write down a recursive formula that the G_m satisfy.

Hint: When a new line is added to the collection $\mathcal{L}$, it must intersect each line already in $\mathcal{L}$ exactly once.

(b) Solve the recursion in (a) to give a closed-form expression for G_m.

Exercise 9. Let G be a finite connected graph. Prove G is a tree if and only if the average degree of a vertex in G is less than 2.

Exercise 10.

Let X be a real $a \times b$ matrix where $a < b$. Find, with proof, $\det(X^T X)$.

Exercise 11. For the 3×3 matrix

$$\mathbf{A} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix},$$

calculate the exponential $e^{t\mathbf{A}}$ for real parameter t.

Exercise 12. A Lévy stable random variable X with parameter $1/2$ and width $b > 0$ has density

$$f_X(x) = \sqrt{\frac{b}{2\pi x^3}} \exp\left(-\frac{b}{2x}\right)$$

for $x > 0$ and zero otherwise. This variable has the moment-generating function

$$M_X(t) = \exp(-\sqrt{-2bt}), \quad t < 0.$$

Determine the probability law of $S_n = (X_1 + \cdots + X_n)/n^2$, where X_k, $k = 1, ..., n$, are i.i.d. Lévy stable variables as above.

Exercise 13.

Let A be an $n \times n$ matrix such that $a_{ik}a_{jk} = a_{kk}a_{ij}$ for all i, j, and k.

(a) Prove that A is symmetric.

(b) Show that if λ is an eigenvalue of A, then $\lambda = 0$ or $\lambda = \operatorname{tr} A$, with $\operatorname{tr}(A) = \sum_{k=1}^{n} a_{kk}$. (Hint: Prove that $A^2 = (\operatorname{tr} A)A$.)

Exercise 14. The amount of time that a certain type of component functions before failing is a random variable X with probability density function

$$f(x) = 3x^2 I_{\{0<x<1\}}.$$

(a) Calculate the expectation EX.

(b) Once the component fails, it is immediately replaced by another one of the same type. If we let X_i denote the lifetime of the ith component to be put in use, then $S_n = \sum_{i=1}^n X_i$ represents the epoch of the nth failure. The long-term rate at which failures occur, call it R, is defined by

$$R := \lim_{n\to\infty} \frac{n}{S_n}.$$

Assuming that the random variables X_i, $i \geq 1$, are independent, show that R is well defined (with probability one) and determine the distribution of the random variable R.

Exercise 15. The joint density function of random variables X and Y is given by

$$f(x, y) = \frac{6}{7}\left(x^2 + \frac{xy}{2}\right)$$

for $0 < x < 1$ and $0 < y < 2$.
(a) Compute the marginal density function of X.
(b) Find $P[X > Y]$.

Exercise 16. When a current I (measured in amperes) flows through a resistance R (measured in ohms), the power generated is given by $W = I^2 R$ (measured in watts). Suppose that I and R are independent random variables with densities

$$f_I(x) = 6x(1-x)I\{0 \leq x \leq 1\}$$

and

$$f_R(y) = 2yI\{0 \leq y \leq 1\}.$$

Determine the distribution function of W.

Exercise 17. Show that the set $\{\ x \in \mathbb{R}^3 : x_3 \geq \sqrt{x_1^2 + x_2^2}\ \}$ is a closed convex cone.

Exercise 18. It is known that $\omega = (-1, 0)$ is a solution to the dual of the following linear program. Find the value of α and a solution of the problem:

$$\begin{array}{lrcl} \min & \alpha x_2 + 5x_4 & & \\ \text{s.t.} & x_1 - 2x_2 - x_4 & = & -4 \\ & x_2 - x_3 + x_4 & = & -1 \\ & x_i & \geq & 0 \quad (i = 1, \cdots, 4) \end{array}$$

Exercise 19.

Define, for $(x, y) \in \mathbb{R}^2$, $(x, y) \neq (0, 0)$:

$$f(x, y) = \frac{2 \sin x \sin y}{x^2 + y^2}$$

a) Show that f is bounded and decide if it can be extended to a continuous function on $\mathbb{R}^2$.

b) Let C be the unit disc: $C = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}$. Compute

$$\int_C f(x, y) dx dy.$$

Exercise 20.

Let f be a continuous function defined on $\mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R}^2$, such that, for all $X, Y \in \mathbb{R}^2$,

$$f(X, Y, Z) = f(AX + T, AY + T, AZ + T)$$

for all $T \in \mathbb{R}^2$ and all 2 by 2 matrices A with determinant equal to 1 or -1.

Show that there exists a function $g : \mathbb{R}^+ \to \mathbb{R}$ such that

$$f(X, Y, Z) = g(A_{XYZ})$$

where $A_{XYZ} = |\det[Y - X, Z - X]|/2$ is the area of the triangle with vertices X, Y, Z. (Hint: Use suitably chosen A and T to reduce X, Y, Z into some canonical position.)
